Name: ____________________ GTID: ____________________

- Fill out your name and Georgia Tech ID number.
- This quiz contains 3 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly.
- Answer the questions in the spaces provided. We will scan the front sides only by default. If you run out of room for an answer, continue on the back of the page and notify the TA when handing in.
- Please write detailed solutions including all steps and computations.
- The duration of the quiz is 30 minutes.

Good luck!

1. Consider the following ODE

$$y' = (y^2 - 4)\sin\left(\frac{\pi}{1+y^2}\right)$$

Determine the equilibrium points of the ODE, classify them, and draw the phase line.

**Solution:** Our equation takes form $y' = f(y)$ with $f(y) = (y^2-4)\sin\left(\frac{\pi}{1+y^2}\right)$ and therefore $y' = 0$ exactly when $f(y) = 0$ which happens either if $y^2 - 4$ or if $\sin\left(\frac{\pi}{1+y^2}\right)$ equals 0. The former occurs for $y = -2$ or $y = 2$, while the latter occurs for $y = 0$.

Therefore the equilibrium points are -2, 2, and 0.

Since $\sin\left(\frac{\pi}{1+y^2}\right)$ is non-negative on $\mathbb{R}$, we see that the sign of $y'$ is equal to the sign of $y^2 - 4$. It follows that $y' < 0$ on $(-2, 0)$ and $(0, 2)$ and $y' > 0$ on $(-\infty, -2)$ and $(2, \infty)$.

Therefore, -2 is a stable equilibrium, 0 is semi-stable, and 2 is unstable.

2. Consider the following ODE
$$(1-t)y'(t) = (y(t))^2.$$
What are the solutions to this ODE? Write your answer as a closed form expression.

**Solution:** After rearranging, we see that this is a separable equation. Therefore, we write
$$\int \frac{1}{y^2} dy = \int \frac{1}{1-t} dt.$$
and solving this integral yields
$$-y^{-1} = -\log|1-t| + c$$
for some $c \in \mathbb{R}$. Rearranging leads to
$$y(t) = \frac{1}{\log|1-t| - c}.$$

3. Consider the differential equation

$$y' = y - y^3.$$

Find the equilibrium points, draw the phase line, and determine whether each equilibrium point is stable, unstable or semi-stable.

**Solution:** The equilibrium points are given by solutions to $y' = 0$. This is the case if and only if $y = -1$, $y = 0$ or $y = 1$. Since the right-hand side

$$f(y) = y - y^3$$

of the differential equation is continuously differentiable, we can use the theorem (linearization about equilibrium points). We calculate

$$f'(y) = 1 - 3y^2.$$

Since $f'(-1) = -2$, $f'(0) = 1$ and $f'(1) = -2$, we get that the equilibrium points are stable, unstable and stable, respectively.

To draw the phase line, note that $y' < 0$ on $(-1, 0) \cup (1, \infty)$ and $y' > 0$ on $(-\infty, -1) \cup (0, 1)$.